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Note

On Finite Projective Planes with a Single (P, l) Transitivity

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A plane of order n having an abelian transitive group of (P, l) transitivities yields generalized Hadamard matrices with entries from a group of order g for any $g \mid n$. Generalized Hadamard matrices of degree 12 are given having entries from groups of order 4 and 3, respectively. Unfortunately these do not yield a plane of order 12 having a (P, l) transitivity. © 1988 Academic Press, Inc.

Let π be a projective plane of order n coordinated by the ternary field T corresponding to some ordered quadrangle of π [1, p. 125]. A ternary operation is defined in T by the equation $y = x \cdot \circ b$. Addition and multiplication are defined in T by

$$a + b = a \cdot 1 \circ b$$

$$ab = a \cdot b \circ 0.$$

T is said to be linear if $x \cdot a \circ b = xa + b$. A linear ternary field is called a cartesian group if its additive loop forms a group. For fixed $a \neq 0$ and $c \neq 0$ it is easily seen that the mapping

$$\sigma_{ac} : b \to c \cdot a \circ b$$

is a fixed point free permutation on the elements of T. If we let P_{ac}

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represent the $n \times n$ permutation matrix corresponding to σ_{ac} we have for fixed c,

$$I_n + \sum P_{ac} = J_n, \tag{1}$$

where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix all of whose entries are 1's. Another result is that for distinct parallel classes having a and d, $a \neq 0 \neq d$, for varying c

$$I_n + \sum P_{ac}^{-1} P_{dc} = J_n. \tag{2}$$

If T is a linear ternary field, the set $\{P_{ac} \mid a, c \in T\}$ forms a transitive, regular permutation group G of order n isomorphic to the group of (L, ∞) elations of π, L the line at infinity [1, p. 130]. The group G is isomorphic to the additive group of T, a cartesian group. Let ρ be a linear representation of G. Let M be the $n \times n$ matrix with $i, 1$ and $1, i$ entries $\rho(I_n)$, $i = 1, \ldots, n$ and i, j entry $\rho(P_{i-1, j-1})$ for $i \neq 1 \neq j$.

Definition. A generalized Hadamard matrix $\mathrm{GH}(qs, K)$ over the group K of order q is an $qs \times qs$ matrix $\mathrm{GH}(qs, K) = (h_{ij})$ such that

(i) $h_{ij} \in K$ for all $1 \leqslant i, j \leqslant qs$,

(ii) $\sum_{m=1}^{qs} h_{im} h_{jm}^{-1} = \sum_{k \in K} sk$ whenever $i \neq j$ where the summation is in the group ring $Z[G]$.

If we take K to be the group generated by the images of the elements of G under ρ, then we see that (1) and (2) imply M is a generalized Hadamard matrix.

All known cartesian groups are elementary abelian and it has been conjectured that all cartesian groups are elementary abelian. This would imply that $n = p^k$ for some integer k and prime p. Another version of the conjecture is that all planes with a transitive group of (L, ∞) elations are of prime power order.

This conjecture is somewhat suspect however. For $n = 12$ there are two generalized Hadamard matrices, one with elements from an elementary abelian four group and the second with elements from a group of order 3. The matrices are given below. The first is due to Seberry [2] and the second is due to the author. The two matrices together do not yield a cartesian group of order 12 but their existence does seem to make the conjecture mentioned above less tenable. Perhaps the reason the matrices do not "fit" together to form a cartesian group is because 12 is too small an order.

Let $\langle a, b \rangle$ be an elementary abelian group of order four, $a^2 = b^2 = 1$,

$ab = ba$. The matrix below is a generalized Hadamard matrix of order 12 with entries in $\langle a, b \rangle$.

$$\begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & b & b & b & ab & ab & ab & a & a & a \\
1 & 1 & 1 & ab & ab & ab & a & a & a & b & b & b \\
1 & b & ab & a & b & ab & b & a & 1 & ab & 1 & a \\
1 & b & ab & b & ab & a & a & 1 & b & 1 & a & ab \\
1 & b & ab & ab & a & b & a & b & a & a & ab & a \\
1 & ab & a & b & a & a & ab & a & b & ab & b & 1 \\
1 & ab & a & a & 1 & b & a & b & ab & b & 1 & ab \\
1 & ab & a & 1 & b & a & b & ab & a & 1 & ab & b \\
1 & a & b & ab & 1 & a & ab & b & 1 & ab & a & b \\
1 & a & b & 1 & a & ab & b & 1 & ab & a & b & ab \\
1 & a & b & a & ab & 1 & 1 & ab & b & b & ab & a
\end{matrix}$$

If c is a cube root of unity the matrix below is a Hadamard matrix of order 12 with entries cube roots of unity.

$$\begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & c & c & c & c & c^2 & c^2 & c^2 & c^2 \\
1 & 1 & c & c^2 & 1 & c & c^2 & c^2 & 1 & c & c & c^2 \\
1 & 1 & c^2 & c & 1 & c^2 & c & c^2 & c & 1 & c^2 & c \\
1 & c & 1 & 1 & c^2 & c^2 & 1 & c^2 & c^2 & c & c & c \\
1 & c & c & c^2 & c^2 & 1 & 1 & c & c & 1 & c^2 & c^2 \\
1 & c & c^2 & c & 1 & 1 & c^2 & c & c^2 & c^2 & c & 1 \\
1 & c & c^2 & c^2 & c^2 & c & c & 1 & 1 & c^2 & 1 & c \\
1 & c^2 & 1 & c & c^2 & c & c^2 & 1 & c & c & c^2 & 1 \\
1 & c^2 & c & 1 & c & 1 & c^2 & c^2 & c & c^2 & 1 & c \\
1 & c^2 & c & c^2 & c & c^2 & c & 1 & c^2 & 1 & c & 1 \\
1 & c^2 & c^2 & c & c & c^2 & 1 & c & 1 & c & 1 & c^2
\end{matrix}$$

References

1. H. P. Dembowski, "Finite Geometries," Springer, Berlin/Heidelberg/New York, 1968.
2. J. Sedberry, "A construction for generalized Hadamard matrices," *J. Statist. Plan. Inference* **4** (1980), 365–368.